# Operator Factorization of Scalar Wave Equation in Frequency-Domian

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*Abstract*— **The partial differential operator factorizations of the scalar wave equation in the time-domain were derived by Engquist and Majda. A set of absorbing boundary conditions were provided by using these equations. An alternative way to derive these equations in the frequency-domain is shown. The limitation and accuracy of the resulting one-way wave equations may be easier to see from this derivation. Recently, the finite-difference vector beam propagation method has been developed. The possibility of the similar finite-difference method based on the one-way wave equations derived by the operator factorization is also discussed.**

THE COMPLETE scalar wave equation in frequency-domain in 3-D Cartesian coordinate may be written as

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\psi=0. \tag{1}$$

A corresponding eigenvalue equation is given by

$$-k_x^2-k_y^2-k_z^2+k^2=0, \tag{2}$$

where vector $\vec{k}=k_x\hat{a}_x+k_y\hat{a}_y+k_z\hat{a}_z$. Eigenvalue equation (2) can be obtained from partial differential equation (1) by transformation: $j(\partial\cdot/\partial x)\rightarrow k_x$, $j(\partial\cdot/\partial y)\rightarrow k_y$ and $j(\partial\cdot/\partial z)\rightarrow k_z$. Eigenvalue equation (2) is consistent with the dispersion relation obtained by using the method of separation of varibles. Considering a wave propagating in the $+z$ direction, component of propagation constant in the $+z$ direction, $k_z$, can be resolved from (2) as

$$k_z=\sqrt{k^2-(k_x^2+k_y^2)}. \tag{3}$$

If the waves satisfy the propagation condition, $k^2>(k^2{}_x+k^2{}_y)$, the Taylor's expansion can be used to expand (3). The series can be truncated in any order: the first-order ($\sqrt{1+x}=1+O(x)$),

$$k_z=k; \tag{4}$$

the second-order ($\sqrt{1+x}=1+\frac{1}{2}x+O(x^2)$),

$$k_z=k\left\{1-\frac{1}{2}\left[\left(\frac{k_x}{k}\right)^2+\left(\frac{k_y}{k}\right)^2\right]\right\}; \tag{5}$$

and the third-order ($\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+O(x^3)$),

$$k_z=k\left\{1-\frac{1}{2}\left[\left(\frac{k_x}{k}\right)^2+\left(\frac{k_y}{k}\right)^2\right]-\frac{1}{8}\left[\left(\frac{k_x}{k}\right)^2+\left(\frac{k_y}{k}\right)^2\right]^2\right\}. \tag{6}$$

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Now, we can obtain the approximate one-way wave equation by conducting the inverse-transformation: $k_x\rightarrow j(\partial\cdot/\partial x)$, $k_y\rightarrow j(\partial\cdot/\partial y)$ and $k_z\rightarrow j(\partial\cdot/\partial z)$, to the Taylor's expansions. We have then the first-order approximate wave equation

$$j\frac{\partial\psi}{\partial z}=k\psi; \tag{7}$$

the second-order approximate wave equation

$$j\frac{\partial\psi}{\partial z}=k\left(1+\frac{1}{2k^2}\frac{\partial^2}{\partial x^2}+\frac{1}{2k^2}\frac{\partial^2}{\partial y^2}\right)\psi; \tag{8}$$

and the third-order approximate wave equation

$$j\frac{\partial\psi}{\partial z}=k\left[1+\frac{1}{2k^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\frac{1}{8k^4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\right]\psi. \tag{9}$$

The Pade's third-order expansion ($\sqrt{1+x}=1+(x)/(2+x/2)+O(x^3)$) of the eigenvalue equation takes the form

$$k_z=k\left\{1-\frac{(\frac{k_x}{k})^2+(\frac{k_y}{k})^2}{2-[(\frac{k_x}{k})^2+(\frac{k_y}{k})^2]/2}\right\}. \tag{10}$$

The corresponding Pade's third-order approximate wave equation is then given by

$$j\frac{\partial\psi}{\partial z}+\frac{j}{4k^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial z}=k\psi+\frac{3}{4k}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right). \tag{11}$$

Equations (7)–(9) and (11) agree with the results by using the differential operator factorization (Engquist and Majda [1]). From the assumption, $k>\sqrt{k_x{}^2+k_y{}^2}$, the propagation waves are approximately governed by these one-way wave equations, while the evenascent waves are excluded from these equations. The well-posedness of these equations can be checked by the standard procedure (Kreiss [3]). The results agree with them obtained by Engquist and Majda [1] in the time-domain. The one-way wave equations in the $-z$ direction can be derived similarly. The accuracy of the equations may be looked by the following example.

### *Numerical Example: Parallel-Plate Waveguide*

For a $\mathrm{TE}_m$ mode, except for the first-order equation the solutions of the above approximate equations and exact wave equation have the form

$$\psi\equiv E_y=\sin\frac{m\pi x}{a}e^{-jk_{zm}z}, \tag{12}$$

where $a$ is the distance between two plates. Inserting this solution into (1), (8), (9), and (11), we have that: the exact full-wave solution has $k_{zm} = \sqrt{k^2 - ((m\pi)/a)^2}$, (here $k_x = (m\pi)/a$ and $k_y = 0$); and the corresponding Taylor's second-order, third-order and Pade's approximations are given by (5), (6) and (10), respectively. These results are expected from the derivation point of view. To give a numerical impression, let $k = 100/m$ and $a = 0.3m$. The total number of the guided modes are nine. For $TE_1$ mode, the relative errors of the propagation constants for the second-order, third-order and Pade's are $1.5 \times 10^{-5}$, $7.7 \times 10^{-8}$ and 0.0, respectively. For $TE_5$ mode, the corresponding errors are $1.3 \times 10^{-2}$, $1.8 \times 10^{-3}$ and $1.0 \times 10^{-3}$. For $TE_9$ mode, the corresponding errors are 0.66, 0.37, and 0.28. The approximations are better for the lower-order modes. The Pade's third-order approximation has the most accurate results to the second-order and third-order Taylor's.

Recently, the finite-difference beam propagation methods have been developed [2], [4]. The possibility of the similar finite-difference method based on the higher order one-way wave equations is then raised here. The first-order equation (7) leads to the geometric optical ray-tracing solution, while the Taylor's third-order equation (9) is ill-posed [1]. The accuracy of the second-order equation (8) is questionable from the above example. Therefore, we may look for the finite-difference schemes for the Pade's third-order equation (11). The von Neumann analysis may be adopted to analyze the stability of the finite-difference schemes for (11). Unfortunately, the explicit schemes are unconditionally unstable. The completely implicit schemes are found unconditionally stable. But, the solution of a strongly stable scheme suffers fast attenuation. Therefore, a usable finite-difference scheme should be neutral-stable or nonattenuative. The amplification factor of the Crank-Nicolson scheme for (11) has been found to be unity. The detailed assessment is beyond the scope of this letter.

To conclude the discussion, the other properties of the higher-order one-way wave equations need to be addressed. The second-order derivative in the expansion direction (+z direction here) has been reduced to the first-order. Therefore, the reflection in the opposite direction, $-z$ direction, is excluded in the higher order equations. To consider the problem completely, the one-way wave equations in the $-z$ direction have to be employed. Fortunately, the higher order one-way wave equations do include the reflection in the other directions. Therefore, the boundary conditions for a metal boundary having the tangential direction parallel to the expansion direction are the same as the full wave equation's, i.e., Maxwell's boundary conditions. If the metal boudary is perpendicular to the $+z$ direction, the equations in the $+z$ direction can be used for the incoming wave utill the boundary, and then, the equations in the $-z$ direction are employed for the reflected wave with the initial value at the boundary provided by the Maxwell's boundary conditions. For a general curved metal boundary, the coordinate rotation of the expansion direction may be used. For optical dielectric waveguides, the first-order absorbing or transparent boundary condition [2] can be posed when a relative large computation window is used.

## REFERENCES

[1] B. Engquist and A. Majda, "Absorbing boundary conditions for the numerical simulation of waves," *Math. Comput.*, vol. 31, pp. 629–651, July 1977.

[2] W. Huang, C. Xu, S. Chu, and S.K. Chaudhuri, "The finite-difference vector beam propagation method: Analysis and assessment," *IEEE J. Lightwave Technol.*, vol. 10, pp. 295–305, Mar. 1992.

[3] H.-O. Kreiss, "Initial boundary value problems for hyperbolic systems," *Commun. Pure Appl. Math.*, vol. 23, pp. 277–298, 1970.

[4] Y. Chung and N. Dagli, "An assessment of finite difference beam propagation method: Application to semiconductor rib waveguide $Y$-junction analysis," *Electron. Lett.*, vol. 26, pp. 711–713, 1990.